

# THE GHOST AND WEAK DIMENSIONS OF RINGS AND RING SPECTRA

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**ABSTRACT.** The primary object of this paper is to prove the conjecture of [HL08a] explaining how to recover the weak dimension of a ring from its derived category. In the process, we develop a theory of weak dimension, which we call ghost dimension, for the generalized rings, known as ring spectra, that arise in algebraic topology.

## INTRODUCTION

In a previous paper [HL08a], the authors considered the problem of recovering the weak dimension of a ring  $R$  from the derived category  $\mathcal{D}(R)$ , together with its distinguished object  $R$ . In that paper, the authors defined the **ghost dimension** of  $R$ ,  $\text{gh.dim. } R$ , and proved that  $\text{gh.dim. } R \geq \text{w.dim. } R$ , with equality holding when  $R$  is coherent or has weak dimension 1. In the present paper, we prove that  $\text{gh.dim. } R = \text{w.dim. } R$  for all rings  $R$ .

The point of doing this, besides its intrinsic interest, is to allow consideration of weak dimension for more general kinds of rings. In algebraic topology, for example, there is a notion of a ring spectrum, or, more precisely, an  $S$ -algebra  $E$  [EKMM97]. Such an  $S$ -algebra has no elements in the usual sense. There is a category of (right)  $E$ -modules, but it is not abelian. However, there is a derived category  $\mathcal{D}(E)$  of  $E$ , and it shares many of the formal properties of the derived category  $\mathcal{D}(R)$  of an ordinary ring  $R$ ; in particular,  $\mathcal{D}(E)$  is a compactly generated triangulated category, and there are derived tensor products and derived Hom objects. In fact, every ring  $R$  has an associated Eilenberg-MacLane  $S$ -algebra  $HR$ , and  $\mathcal{D}(HR)$  is equivalent to  $\mathcal{D}(R)$ . To define invariants of such  $S$ -algebras  $E$ , then, one way to proceed is to define usual ring invariants, such as the weak or global dimension, in terms of  $\mathcal{D}(R)$ , and then apply this definition to  $\mathcal{D}(E)$  as well.

The second author did this for the (right) global dimension in his thesis, and we now summarize this. For further details, see [HL08a]. Define a map  $f: X \rightarrow Y$  in  $\mathcal{D}(E)$  to be a **ghost** if  $\mathcal{D}(E)(E, f)_* = 0$ . In the case that  $E$  is an ordinary ring  $R$ , a ghost is then just a map that induces the zero homomorphism on homology. If  $E$  is an  $S$ -algebra, a ghost is a map that induces the zero homomorphism on homotopy groups. The second author shows that the right global dimension of a ring  $R$  is the least  $n$  for which every composite of  $n + 1$  ghosts in  $\mathcal{D}(R)$  is null, or  $\infty$  if there are arbitrarily long nonzero composites of ghosts. We can then define the global dimension of an  $S$ -algebra  $E$  in an analogous fashion. The authors did this and investigated the  $S$ -algebras of global dimension 0 in [HL08b].

Weak dimension is more complicated, and there seem to be many possible definitions. A major goal of this paper is to elucidate the different possibilities and to

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find the correct one. The **ghost dimension** of an  $S$ -algebra  $E$  or a ring  $R$  is the least  $n$  such that every composite

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n+1}} X_{n+1}$$

of  $n + 1$  ghosts in  $\mathcal{D}(E)$  (or  $\mathcal{D}(R)$ ), where  $X_0$  is a compact object, is null (or  $\infty$  if there are arbitrarily long nonzero composites of ghosts out of compact objects). Recall that  $X$  is compact in a triangulated category  $\mathcal{C}$  if the functor  $\mathcal{C}(X, -)$  preserves coproducts. In particular, the compact objects of  $\mathcal{D}(R)$  are the perfect complexes (complexes quasi-isomorphic to bounded complexes of finitely generated projectives), and the compact objects of  $\mathcal{D}(E)$  are the retracts of finite cell  $E$ -modules. The ghost dimension of a ring was discussed in [HL08a], as mentioned above.

In addition, a version of weak dimension closely related to Rouquier's definition of the dimension of a triangulated category [Rou08] is defined in [HL08b]. Neither of these uses the notion of a flat  $E$ -module. This obvious oversight was made because of a difficulty with flat modules that we now recall. If  $E$  is an  $S$ -algebra, and  $F_*$  is a flat left  $E_* = \pi_* E = \mathcal{D}(E)(E, E)_*$ -module, then we can form a homology theory (a coproduct-preserving exact functor to abelian groups) on  $\mathcal{D}(E)$  that takes  $M$  to  $M_* \otimes_{E_*} F_*$ . One would like to say that Brown representability for homology theories then forces there to be a left  $E$ -module  $F$  with  $\pi_* F \cong F_*$ . Unfortunately, Brown representability does not hold for a general ring spectrum [Nee97, CKN01], so flat modules may not be realizable. This worrying phenomenon led us to doubt the utility of flat modules. However, we use them in this paper. One of the surprising things we discover is the following. Call  $X \in \mathcal{D}(E)$  **flat** if  $X_*$  is a flat  $E_*$ -module. Then we prove that  $X$  is flat if and only if every ghost  $f$  whose domain is  $X$  is phantom, in the sense that  $\mathcal{D}(E)(A, f) = 0$  for all compact  $A \in \mathcal{D}(E)$ . This gives us two different notions of flat dimension. The one most similar to the algebraic situation we call the **constructible flat dimension**,  $\text{con. flat dim. } X$ . It is a measure of how many steps one needs to construct  $X$  from flat objects of  $\mathcal{D}(E)$ . We reserve the term **flat dimension**,  $\text{flat dim. } X$ , for the smallest  $n$  such that every composite of  $n + 1$  ghosts with domain  $X$  is phantom. This seems algebraically strange, but has better properties. This gives several more notions of weak dimension: the maximal constructible flat (resp. flat) dimension of a compact  $E$ -module, and the maximal constructible flat (resp. flat) dimension of an arbitrary  $E$ -module. We show that all of these are equal to the ghost dimension, except possibly the maximal constructible flat dimension of an arbitrary  $E$ -module. The conjecture of [HL08a] that  $\text{gh. dim. } R = \text{w. dim. } R$  then follows.

In the end, we are left with three definitions of weak dimension for an  $S$ -algebra  $E$ . There is  $\text{gh. dim. } E$ , which coincides with the maximal flat dimension of any object. There is the maximal constructible flat dimension of any object, which agrees with  $\text{gh. dim. } E$  for  $E = HR$ , but possibly not in general. And there is the Rouquier dimension  $\text{Rouq. dim. } E$ , which agrees with  $\text{gh. dim. } E$  when  $E_*$  is coherent. We prove that the ghost dimension is right-left symmetric, which we have been unable to do with any of the other definitions. Hence we argue that the ghost dimension is the proper version of weak dimension for  $S$ -algebras  $E$ .

This subject sorely needs examples, in order to be sure that all these definitions are in fact distinct. It should be possible to find an ordinary ring  $R$  such that the Rouquier dimension of  $\mathcal{D}(R)$  is distinct from the other dimensions. Such an example would involve serious analysis of the derived category of a non-coherent

ring. To determine whether the constructible flat dimension is different from the flat dimension would seem to require a new idea.

Note that all modules we use in this paper are right modules unless explicitly stated otherwise. The reader who is interested only in ordinary rings can read  $R$  everywhere the symbol  $E$  or  $E_*$  appears, read “chain complex of  $R$ -modules” whenever the term “ $E$ -module” appears, and read  $H_*X$  everywhere  $X_*$  appears, for  $X$  an  $E$ -module.

## 1. GHOST DIMENSION AND ROUQUIER DIMENSION

For an  $S$ -algebra  $E$  or a ring  $R$ , the authors have previously considered two different possible definitions related to weak dimension, which we now discuss. First of all, we can define the Rouquier dimension to be the maximum number of steps needed to build a compact object of  $\mathcal{D}(E)$  from finitely many copies of  $E$  (along the lines of Rouquier [Rou08]). In more detail, given a class  $\mathcal{A}$  of objects of  $\mathcal{D}(E)$ , define  $\langle \mathcal{A} \rangle^n$  inductively as follows. Define  $\langle \mathcal{A} \rangle^0$  to be the collection of all retracts of coproducts of suspensions of elements of  $\mathcal{A}$ , and define an object  $X$  to be in  $\langle \mathcal{A} \rangle^n$  if and only if it is a retract of an object  $\tilde{X}$  for which there is an exact triangle

$$A \rightarrow Y \rightarrow \tilde{X} \rightarrow A$$

where  $A \in \langle \mathcal{A} \rangle^0$ , and  $Y \in \langle \mathcal{A} \rangle^{n-1}$ . If  $\mathcal{A}$  is a class of compact objects, we define  $\langle \mathcal{A} \rangle_f^n$  similarly, with  $\langle \mathcal{A} \rangle_f^0$  being the collection of all retracts of *finite* coproducts of suspensions of elements of  $\mathcal{A}$ , and then using the same induction procedure to define  $\langle \mathcal{A} \rangle_f^n$ . Then  $\langle \mathcal{A} \rangle_f^n$  consists of compact objects in  $\langle \mathcal{A} \rangle^n$ , but there may be compact objects in  $\langle \mathcal{A} \rangle^n$  that are nevertheless not in  $\langle \mathcal{A} \rangle_f^n$ .

We define the **Rouquier dimension** of  $E$  (or  $R$ ),  $\text{Rouq. dim. } E$ , to be the smallest  $n$  such that  $\langle E \rangle_f^n$  is all of the compact objects, or  $\infty$  if no such  $n$  exists. This was called the weak dimension in [HL08b], but that seems inappropriate, since we do not know that it agrees with the weak dimension when  $E$  is an ordinary ring  $R$ . We define the **ghost dimension** of  $E$  (or  $R$ ),  $\text{gh. dim. } E$ , to be the smallest  $n$  such that  $\langle E \rangle^n$  contains all the compact objects. We also define the **projective dimension**,  $\text{proj. dim. } X$ , of a given object  $X$  to be the smallest  $n$  such that  $X \in \langle E \rangle^n$ . This was called the **ghost length** in [HL08a]. Then  $\text{gh. dim. } E$  is the supremum of  $\text{proj. dim. } X$  for  $X$  compact.

The following proposition explains the connection to the definition given in the introduction.

**Proposition 1.1.** *Suppose  $E$  is an  $S$ -algebra or an ordinary ring, and  $X \in \mathcal{D}(E)$ . Then  $\text{proj. dim. } X \leq n$  if and only if every composite of  $n + 1$  ghosts with domain  $X$  is the zero map. Furthermore,  $\text{proj. dim. } X \leq \text{proj. dim.}_{E_*} X_*$ , with equality when  $\text{proj. dim. } X = 0$  and also when  $E$  is an ordinary ring and  $X$  is the projective resolution of a module  $M$ .*

This proposition is the content of Proposition 1.1, the proof of Proposition 1.3, and Lemma 1.4 of [HL08a], although Proposition 1.1 of [HL08a] is really due to Christensen [Chr98, Theorem 3.5].

We commonly call the objects  $P$  with  $\text{proj. dim. } P = 0$  **projective**, as this proposition implies  $P$  is projective if and only if  $P_*$  is a projective  $E_*$ -module. We note that the universal coefficient spectral sequence of [EKMM97, Theorem IV.4.1]

implies that if  $P$  is projective then the natural map

$$\mathcal{D}(E)(P, X) \rightarrow \mathrm{Hom}_{E_*}(P_*, X_*)$$

is an isomorphism for all  $X \in \mathcal{D}(E)$ . The converse is also true, for if this natural map is an isomorphism, then there are no nonzero ghosts with domain  $P$ .

The following lemma gives the most obvious relationship between ghost dimension and Rouquier dimension.

**Lemma 1.2.** *Suppose  $E$  is an  $S$ -algebra or an ordinary ring. Then*

$$\mathrm{gh. dim. } E \leq \mathrm{Rouq. dim. } E,$$

*with equality holding when  $\mathrm{gh. dim. } E = 0$ .*

*Proof.* The inequality is clear. If  $\mathrm{gh. dim. } E = 0$ , then every compact object is a retract of a coproduct of suspensions of  $E$ , so is also a retract of a finite coproduct of suspensions of  $E$ .  $\square$

Note that  $S$ -algebras with  $\mathrm{gh. dim. } E = 0$  are called **von Neumann regular**, because if  $R$  is an ordinary ring,  $\mathrm{gh. dim. } R = 0$  if and only if  $R$  is von Neumann regular (see [HL08b]).

The ghost dimension and the Rouquier dimension agree when  $E_*$  is coherent, as we see in the following proposition.

**Proposition 1.3.** *Suppose  $E$  is an  $S$ -algebra or an ordinary ring for which  $E_*$  is coherent. Then  $X \in \langle E \rangle_f^n$  if and only if  $X \in \langle E \rangle^n$  and  $X$  is compact. Thus  $\mathrm{gh. dim. } E = \mathrm{Rouq. dim. } E$ .*

There is no reason to think that  $\mathrm{gh. dim. } E = \mathrm{Rouq. dim. } E$  if  $E_*$  is not coherent, even if  $E$  is an ordinary ring  $R$ , but we do not know a counterexample.

*Proof.* We first prove the well-known fact that, since  $E_*$  is coherent, for every compact object  $X$  of  $\mathcal{D}(E)$ ,  $X_*$  is a finitely presented  $E_*$ -module. Consider the class  $\mathcal{C}$  of  $X$  for which  $X_*$  is a finitely presented  $E_*$ -module. We claim that  $\mathcal{C}$  is a thick subcategory. Given this, since  $\mathcal{C}$  contains  $E$ , it contains all the compact objects as required (this is well-known; see [HPS97, Theorem 2.1.3] for a general approach to this fact). To prove that  $\mathcal{C}$  is thick, we must show that it is closed under suspensions and retracts, which is obvious in this case, and also if we have an exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

in which  $X, Y$  are in  $\mathcal{C}$ , then  $Z \in \mathcal{C}$ . Given such an exact triangle,  $Z_*$  is an extension of  $\mathrm{coker } f_*$  by  $\ker(\Sigma f_*)$ . Finitely presented modules are always closed under cokernels of morphisms and extensions [Lam99, Lemma 4.54]. If  $E_*$  is coherent, finitely presented modules are closed under kernels of morphisms as well, and so  $Z_*$  is finitely presented. Indeed, if  $g: M \rightarrow N$  is a morphism of finitely presented modules over a coherent ring, then  $M/\ker f \cong \mathrm{im } f$  is a finitely generated submodule of the finitely presented module  $N$ , so is finitely presented. Hence  $\ker f$  must be a finitely generated submodule of the finitely presented module  $M$ , so is finitely presented.

Now suppose  $X \in \langle E \rangle^n$  and  $X$  is compact. By induction, because  $X_*$  is finitely presented, we can choose finite coproducts  $P_i$  of suspensions of  $E$  and exact triangles

$$X_{i+1} \rightarrow P_i \xrightarrow{f_i} X_i \xrightarrow{\delta_i} \Sigma X_{i+1}$$

where  $X_0 = X$  and  $f_i$  is onto on homotopy, so  $\delta_i$  is a ghost. Of course each  $X_i$  is compact. Then consider the exact triangle

$$X_{i+1} \rightarrow Y_i \rightarrow X \rightarrow \Sigma^{i+1} X_{i+1}.$$

By using the  $3 \times 3$  lemma (well-known, but stated in [HPS97, Lemma A.1.2]) on the square

$$\begin{array}{ccc} X & \longrightarrow & \Sigma^i X_i \\ \parallel & & \downarrow \\ X & \longrightarrow & \Sigma^{i+1} X_{i+1} \end{array}$$

we see that there is an exact triangle

$$\Sigma^{i-1} P_i \rightarrow Y_{i-1} \rightarrow Y_i \rightarrow \Sigma^i P_i.$$

Hence  $Y_i$  is in  $\langle E \rangle_f^i$ . On the other hand, the map  $X \rightarrow \Sigma^{n+1} X_{n+1}$  is the composite of  $n+1$  ghosts, so it is null since  $X \in \langle E \rangle^n$ , using Proposition 1.1. Hence  $X$  is a retract of  $Y_n$ , so  $X \in \langle E \rangle_f^n$ .  $\square$

## 2. FLAT DIMENSION

We now offer a different approach to the weak dimension of an  $S$ -algebra or an ordinary ring using flat modules. As discussed in the introduction, we did not use these in [HL08a] because of the fundamental issue that homology functors are not always representable in  $\mathcal{D}(E)$  for an  $S$ -algebra  $E$ , or even a ring  $R$ .

However, we can still define  $\mathcal{F}$  to be the class of objects  $F$  in  $\mathcal{D}(E)$  such that  $F_*$  is flat over  $E_*$ . In this case, we say that  $F$  is **flat** (as an object of  $\mathcal{D}(E)$ ). We can then define an  $E$ -module  $X \in \mathcal{D}(E)$  to have **constructible flat dimension**  $n$ , written  $\text{con. flat dim. } X = n$ , if  $X \in \langle \mathcal{F} \rangle^n$ . Note that  $\text{con. flat dim. } X \leq \text{proj. dim. } X$ , since every projective is flat. We can then consider the maximal constructible flat dimension of any object in  $\mathcal{D}(R)$ , or of just a compact object in  $\mathcal{D}(R)$ . Both of these are possible candidates for something like weak dimension. In principle, we could also consider a definition similar to the Rouquier dimension, using compact flat objects to resolve arbitrary compact objects, but we will see that a compact flat object is projective, so this would just recover Rouquier dimension.

**Proposition 2.1.** *We have  $\text{con. flat dim. } X \leq \text{flat dim. } X_*$  for all  $X \in \mathcal{D}(E)$ . In particular, the maximal constructible flat dimension of an object in  $\mathcal{D}(E)$  is bounded above by  $\text{w. dim. } E_*$ .*

*Proof.* There is nothing to prove if  $\text{flat dim. } X_*$  is infinite, so suppose  $\text{flat dim. } X_* = n$ . Then by beginning a projective resolution of  $X_*$ , we get an exact sequence of  $E_*$ -modules

$$0 \rightarrow F \rightarrow P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} X_* \rightarrow 0$$

where each  $P_i$  is projective over  $E_*$  and  $F$  is flat over  $E_*$ . This gives us short exact sequences

$$0 \rightarrow K_{i+1} \rightarrow P_i \rightarrow K_i \rightarrow 0$$

for  $i \leq n-1$ , where  $K_i = \ker d_{i-1}$ ,  $K_0 = X_*$ , and  $K_n = F$ . Because the  $P_i$  are projective, these short exact sequences are uniquely realizable by exact triangles in  $\mathcal{D}(E)$

$$X_{i+1} \rightarrow Q_i \rightarrow X_i \rightarrow \Sigma X_{i+1}$$

where  $X_0 = X$ ,  $(X_i)_* = K_i$  and  $(Q_i)_* = P_i$ . In more detail,  $P_i$  is a retract of a direct sum of copies of  $E_*$ . Thus we can let  $Q_i$  be the corresponding retract of a coproduct of copies of  $E$ . Then one checks that a map out of  $Q_i$  to any object  $Y$  is equivalent to a map  $P_i \rightarrow Y_*$ . This gives us exact triangles of the form

$$\Sigma^{i-1}X_i \rightarrow Y_i \rightarrow X \rightarrow \Sigma^i X_i$$

for all  $i$ , where the last map is the composite of the maps  $\Sigma^j X_j \rightarrow \Sigma^{j+1} X_{j+1}$ . Using the  $3 \times 3$  lemma on the commutative square

$$\begin{array}{ccc} X & \longrightarrow & \Sigma^i X_i \\ \parallel & & \downarrow \\ X & \longrightarrow & \Sigma^{i+1} X_{i+1} \end{array}$$

gives us exact triangles

$$\Sigma^{i-1}Q_i \rightarrow Y_i \rightarrow Y_{i+1} \rightarrow \Sigma^i Q_i$$

for all  $i$ . In particular,  $\text{con. flat dim. } Y_i \leq i - 1$ . Now the exact triangle

$$\Sigma^{n-1}X_n \rightarrow Y_n \rightarrow X \rightarrow \Sigma^n X_n$$

shows that  $\text{con. flat dim. } X \leq n$ , since  $(X_n)_*$  is flat.  $\square$

We now give an alternative characterization of the flat objects in  $\mathcal{D}(E)$ . Recall that a **phantom** map in  $\mathcal{D}(E)$  is a map  $f: X \rightarrow Y$  such that  $fg = 0$  for all  $g: A \rightarrow X$  where  $A$  is compact. We need the following lemma.

**Lemma 2.2.** *Suppose  $E$  is an  $S$ -algebra. A map  $f: X \rightarrow Y$  is phantom in  $\mathcal{D}(E)$  if and only if  $\pi_*(f \wedge_E Z) = 0$  for all left  $E$ -modules  $Z$  if and only if  $\pi_*(f \wedge_E Z) = 0$  for all compact left  $E$ -modules  $Z$ .*

*Proof.* Spanier-Whitehead duality implies that  $\pi_*(f \wedge_E Z) = 0$  for all compact left  $E$ -modules  $Z$  if and only if  $\mathcal{D}(E)(W, f) = 0$  for all compact right  $E$ -modules  $W$ , which of course is the definition of  $f$  being phantom. It remains to show that, under this condition,  $\pi_*(f \wedge_E Z) = 0$  for all left  $E$ -modules  $Z$ . But  $\pi_*(- \wedge_E Z)$  is a homology theory on  $\mathcal{D}(E)$ , and phantom maps vanish on all homology theories [CS98, Proposition 1.1]. The reader should note that Christensen and Strickland are working in the ordinary category of spectra, where Spanier-Whitehead duality is internal, but the same proof will work for a general  $S$ -algebra  $E$  as long as we remember that Spanier-Whitehead duality shifts from left to right  $E$ -modules and vice versa.  $\square$

**Proposition 2.3.** *Suppose  $E$  is an  $S$ -algebra or a ring, and  $X \in \mathcal{D}(E)$ . Then the following are equivalent:*

- (1)  $X_*$  is a flat  $E_*$ -module.
- (2) Every ghost with domain  $X$  is phantom.
- (3) There is an exact triangle

$$P \rightarrow X \xrightarrow{g} Y \rightarrow \Sigma P$$

where  $P$  is projective and  $g$  is phantom.

- (4) Every map  $A \rightarrow X$ , where  $A$  is compact, factors through a compact projective object.

(5) *The natural map*

$$X_* \otimes_{E_*} Z_* \rightarrow \pi_*(X \wedge_E Z)$$

*is an isomorphism for all left  $E$ -modules  $Z$ .*

If  $E$  is an ordinary ring  $R$ , then  $X \wedge_E Z$  would be the total left derived tensor product of the chain complex  $X$  of right  $R$ -modules and the chain complex  $Z$  of left  $R$ -modules.

*Proof.* For any  $X$ , there is an exact triangle

$$P \rightarrow X \xrightarrow{g} Y \rightarrow \Sigma P$$

in which  $P$  is projective and  $g$  is a ghost. Indeed, we simply take an epimorphism from a free  $E_*$ -module  $P_*$  to  $X_*$ . We then let  $P$  be the corresponding coproduct of suspensions of  $E$ , which is projective, and realize the map  $P_* \rightarrow X_*$  as a map  $P \rightarrow X$ . The cofiber  $g$  is then automatically a ghost, and every other ghost with domain  $X$  factors through  $g$ .

It follows from this that part (2) and (3) are equivalent. This also means that part (3) implies part (4), since part (3) means that a map  $A \rightarrow X$ , where  $A$  is compact, factors through a projective, and therefore a free  $E$ -module. Since  $A$  is compact, it must factor through a finite coproduct of suspensions of  $E$ .

We now show that part (4) implies part (1). Recall the filtered category  $\Lambda(X)$  from [HPS97, Section 2.3] of maps from a compact object into  $X$ , and consider the full subcategory  $\Lambda'(X)$  of maps from a compact projective into  $X$ . Given part (4),  $\Lambda'(X)$  is cofinal in  $\Lambda(X)$  and itself filtered. Thus, for any homology theory  $H$ ,  $H(X) = \operatorname{colim}_{\Lambda'(X)} H(P_\alpha)$  by [HPS97, Corollary 2.3.11]. In particular,  $X_*$  is a colimit of finitely generated projective modules, so is flat.

To see that part (1) implies part (5), use the universal coefficient spectral sequence

$$\operatorname{Tor}_{s,t}^{E_*}(X_*, Z_*) \Rightarrow \pi_{t-s}(X \wedge_E Z)$$

of [EKMM97, Theorem IV.4.1].

To see that part (5) implies part (2), suppose that  $g$  is a ghost with domain  $X$ . Part (5) then implies that  $\pi_*(g \wedge_E Z) = 0$  for all left  $E$ -modules  $Z$ , which implies that  $g$  is phantom by Lemma 2.2.  $\square$

Recall that, for a general ring  $R$ , there are finitely generated flat modules which are not projective, though of course every finitely presented flat module is projective. The following corollary is an analog of this fact for  $S$ -algebras.

**Corollary 2.4.** *Suppose  $E$  is an  $S$ -algebra or an ordinary ring. If  $X$  is a compact flat object of  $\mathcal{D}(E)$ , then  $X$  is projective.*

*Proof.* The universal ghost out of  $X$  is phantom, and hence null. Thus  $X$  is projective, necessarily finitely generated since  $X$  is compact.  $\square$

We have been unable to fully generalize Proposition 2.3 to objects  $X$  with  $\operatorname{con. flat dim.} X = n$ . We therefore make the following definition.

**Definition 2.5.** Suppose  $E$  is an  $S$ -algebra or an ordinary ring, and  $X \in \mathcal{D}(E)$ . We say that  $X$  has **flat dimension** at most  $n$ , written  $\operatorname{flat dim.} X \leq n$ , if every composite of  $n + 1$  ghosts with domain  $X$  is phantom.

We then have the following theorem.

**Theorem 2.6.** *Suppose  $E$  is an  $S$ -algebra or an ordinary ring, and  $X \in \mathcal{D}(E)$ . Then  $\text{flat dim. } X \leq \text{con. flat dim. } X$ , and the following are equivalent.*

- (1)  $\text{flat dim. } X \leq n$ .
- (2) *There is an exact triangle*

$$B \rightarrow X \xrightarrow{g} Y \rightarrow \Sigma B$$

*where  $\text{proj. dim. } B \leq n$  and  $g$  is phantom.*

- (3) *Every map  $A \rightarrow X$ , where  $A$  is compact, factors through a compact object  $B$  with  $\text{proj. dim. } B \leq n$ .*
- (4) *For any left  $E$ -module  $Z$ , in the universal coefficient spectral sequence*

$$E_{s,t}^2 = \text{Tor}_{s,t}^{E_*}(X_*, Z_*) \Rightarrow \pi_{t-s}(X \wedge_E Z),$$

*we have  $E_{s,*}^\infty = 0$  for all  $s > n$ .*

It would be good to find an example where  $\text{flat dim. } X < \text{con. flat dim. } X$ , or to prove they are always equal.

*Proof.* We show that  $\text{con. flat dim. } X \leq n$  implies every composition of  $n+1$  ghosts out of  $X$  is phantom by induction on  $n$ . The base case of  $n = 0$  is Proposition 2.3. For the induction step, suppose  $\text{con. flat dim. } X \leq n$ ,

$$X \xrightarrow{g_1} Z_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n+1}} Z_{n+1}$$

is the composition  $g$  of  $n+1$  ghosts, and  $f: A \rightarrow X$  is a map from a compact object. We must show  $gf = 0$ . Since  $\text{con. flat dim. } X \leq n$ , there is a cofiber sequence

$$F \rightarrow Y \xrightarrow{s} \tilde{X} \xrightarrow{h} \Sigma F$$

where  $F$  is flat,  $\text{con. flat dim. } Y \leq n-1$ , and there are maps

$$X \xrightarrow{i} \tilde{X} \xrightarrow{r} X$$

with  $ri = 1_X$ . Since  $gf = (gr)(if)$ , and  $gr$  is again a composition of  $n+1$  ghosts, we can assume  $X = \tilde{X}$ .

The composition  $hf$  factors through a finitely generated projective  $P$ , by Proposition 2.3. This gives us a commutative diagram

$$\begin{array}{ccccccc} \Sigma^{-1}P & \longrightarrow & B & \xrightarrow{t} & A & \xrightarrow{u} & P \\ \downarrow & & f' \downarrow & & f \downarrow & & \downarrow \\ F & \longrightarrow & Y & \xrightarrow{s} & X & \xrightarrow{h} & \Sigma F \end{array}$$

whose rows are exact triangles. Note that  $B$  is necessarily a compact object, and so the composition  $g_n \circ \cdots \circ g_1 s f'$  is null, since  $\text{flat dim. } Y \leq n-1$ . Hence we have

$$g_n \circ \cdots \circ g_1 f t = 0 \text{ so } g_n \circ \cdots \circ g_1 f = v u$$

for some map  $v: P \rightarrow Z_n$ . But then  $g_{n+1}v = 0$ , since  $P$  is projective, and so

$$g_{n+1} \circ \cdots \circ g_1 f = 0$$

as required. This completes the proof that  $\text{flat dim. } X \leq \text{con. flat dim. } X$ .

The work of Christensen [Chr98, Theorem 3.5] implies that, for any  $X$ , there is an exact triangle

$$B \rightarrow X \xrightarrow{g} Y \rightarrow \Sigma B$$



with  $\text{proj. dim. } B \leq n$  and  $g$  is a composite of  $n+1$  ghosts. But then every composite of  $n+1$  ghosts with domain  $X$  factors through  $g$ . Thus every composite of  $n+1$  ghosts is phantom if and only if  $g$  is phantom, so part (1) and part (2) are equivalent.

Now, the universal coefficient spectral sequence for  $\pi_*(X \wedge_E Z)$  is constructed as follows. Beginning with  $X_0 = X$ , we construct exact triangles

$$X_{i+1} \rightarrow Q_i \xrightarrow{h_i} X_i \xrightarrow{k_i} \Sigma X_{i+1}$$

as in the proof of Proposition 2.1, in which  $Q_i$  is projective,  $h_i$  is onto on homotopy, and  $k_i$  is a ghost. We then smash them with  $Z$  and take homotopy to get our spectral sequence of homological type. An element in  $\pi_*(X \wedge_E Z)$  is detected in  $E_{s,*}^\infty$  if and only if it is in the kernel of

$$\pi_*(X \wedge_E Z) \rightarrow \pi_*(\Sigma^j X_j \wedge_E Z)$$

for  $j = s$  but not for  $j = s - 1$ . So we must determine when the map

$$\pi_*(X \wedge_E Z) \rightarrow \pi_*(\Sigma^n X_n \wedge_E Z)$$

is zero for all  $Z$ . However, comparison of this construction of [EKMM97, Section IV.5] with [Chr98, Theorem 3.5] shows that in fact the map  $X \rightarrow \Sigma^n X_n$  is the same as the universal composite of  $n+1$  ghosts out of  $X$ ,  $g: X \rightarrow Y$  of the previous paragraph. Lemma 2.2 then shows part (2) and part (4) are equivalent.

It is clear that if every map from a compact to  $X$  factors through a  $Y$ , compact or not, with  $\text{proj. dim. } Y \leq n$ , then every composite of  $n+1$  ghosts out of  $X$  is phantom, so part (3) implies part (1). For the converse, in view of part (2), it suffices to prove that every map from a compact object to an  $A$  with  $\text{proj. dim. } A \leq n$  factors through a compact  $B$  with  $\text{proj. dim. } B \leq n$ . We proceed by induction on  $n$ . The base case  $n = 0$  is implied by Proposition 2.3. For the induction step, suppose we have a map  $f: F \rightarrow A$ , where  $F$  is compact and  $\text{proj. dim. } A \leq n+1$ . Then we have an exact triangle

$$C \xrightarrow{r} P \xrightarrow{s} \tilde{A} \xrightarrow{t} \Sigma C$$

with  $P$  projective,  $\text{proj. dim. } C \leq n$ , and  $A$  is a retract of  $\tilde{A}$ . It suffices to show that the composite

$$F \rightarrow A \rightarrow \tilde{A}$$

factors through a compact  $B$  with  $\text{proj. dim. } B \leq n+1$ . We can therefore assume  $A = \tilde{A}$ . By the induction hypothesis, we can write  $tf = \phi h$  for some  $h: F \rightarrow D$ , where  $D$  is compact with  $\text{proj. dim. } D \leq n$ . This gives us a map of exact triangles

$$\begin{array}{ccccccc} \Sigma^{-1}D & \longrightarrow & Z & \longrightarrow & F & \xrightarrow{h} & D \\ \Sigma^{-1}\phi \downarrow & & \psi \downarrow & & f \downarrow & & \downarrow \phi \\ C & \xrightarrow{r} & P & \xrightarrow{s} & A & \xrightarrow{t} & \Sigma C \end{array}$$

But then  $Z$  is necessarily compact, so we can write  $\psi = j\tau$ , where the codomain  $Q$  of  $\tau$  is compact projective. By taking the weak pushout (which amounts to applying

the  $3 \times 3$  lemma), we get the following diagram, whose rows are exact triangles.

$$\begin{array}{ccccccc}
 \Sigma^{-1}D & \longrightarrow & Z & \longrightarrow & F & \xrightarrow{h} & D \\
 \parallel & & \tau \downarrow & & \sigma \downarrow & & \parallel \\
 \Sigma^{-1}D & \longrightarrow & Q & \longrightarrow & B' & \longrightarrow & D \\
 \Sigma^{-1}\phi \downarrow & & j \downarrow & & \rho \downarrow & & \downarrow \phi \\
 C & \xrightarrow{r} & P & \xrightarrow{s} & A & \xrightarrow{t} & \Sigma C
 \end{array}$$

Now  $B'$  is a compact object with  $\text{proj. dim. } B' \leq n+1$ , but unfortunately  $\rho\sigma$  need not be equal to  $f$ . Nevertheless, we do have

$$t\rho\sigma = \phi h = tf,$$

so  $f - \rho\sigma = sq$  for some map  $q: F \rightarrow P$ . But then  $q = iq'$  for some map  $q': F \rightarrow Q'$ , where  $Q'$  is compact projective. Altogether then, we have

$$f = \rho\sigma + siq'.$$

This means that  $f$  factors through the compact object  $B' \amalg Q'$ . Indeed,  $f$  is the composite

$$F \xrightarrow{(\sigma, q')} B' \amalg Q' \xrightarrow{\rho + si} A.$$

Since  $\text{proj. dim.}(B' \amalg Q') \leq n+1$ , the proof is complete.  $\square$

**Corollary 2.7.** *Suppose  $E$  is an  $S$ -algebra or an ordinary ring. If  $X$  is a compact object of  $\mathcal{D}(E)$ , then*

$$\text{flat dim. } X = \text{con. flat dim. } X = \text{proj. dim. } X.$$

*In particular,  $\text{gh. dim. } E$  is the maximal (constructible or not) flat dimension of a compact object of  $\mathcal{D}(E)$ , or  $\infty$  if there is no such maximal dimension.*

*Proof.* We always have  $\text{flat dim. } X \leq \text{con. flat dim. } X \leq \text{proj. dim. } X$ . Suppose  $\text{flat dim. } X = n$  and  $X$  is compact. Then every composition of  $n+1$  ghosts out of  $X$  is phantom, hence null. Thus  $\text{proj. dim. } X \leq n$ , so  $\text{proj. dim. } X = \text{flat dim. } X$ .  $\square$

**Corollary 2.8.** *Suppose  $E$  is an  $S$ -algebra or an ordinary ring. Then  $\text{gh. dim. } E = \text{sup flat dim. } X$  as  $X$  runs through arbitrary objects of  $\mathcal{D}(E)$ .*

We do not know whether this corollary remains true for the constructible flat dimension.

*Proof.* Corollary 2.7 implies that

$$\text{gh. dim. } E \leq \sup_X \text{flat dim. } X.$$

Now suppose  $\text{gh. dim. } E = n$ , so that  $\text{flat dim. } F \leq n$  for all compact objects  $F$  of  $\mathcal{D}(E)$ . Choose an arbitrary  $X \in \mathcal{D}(E)$ , and consider the universal coefficient spectral sequence

$$E_{s,t}^2 = \text{Tor}_{s,t}^{E_*}(X_*, Z_*) \Rightarrow \pi_{t-s}(X \wedge_E Z)$$

for an arbitrary left  $E$ -module  $Z$ . We must show that  $E_{s,*}^\infty = 0$  for  $s > n$ . Since this spectral sequence is of homological type, this means we must show that every

element in  $\pi_*(X \wedge_E Z)$  is in filtration  $n$  (and possibly lower filtration as well). But the functor  $\pi_*(- \wedge_E Z)$  is a homology theory on right  $E$ -modules, so

$$\pi_*(X \wedge_E Z) = \operatorname{colim} \pi_*(F \wedge_E Z),$$

where the colimit is taken over all maps  $F \rightarrow X$  from a compact object to  $X$ . Then any element of  $\pi_*(X \wedge_E Z)$  comes from some  $\pi_*(F \wedge_E Z)$ , where it lies in filtration  $n$ . Naturality of the spectral sequence implies that it also lies in filtration  $n$  in  $\pi_*(X \wedge_E Z)$ .  $\square$

We now point out another advantage of the ghost dimension; it is left-right symmetric, like the usual weak dimension of rings.

**Theorem 2.9.** *Suppose  $E$  is an  $S$ -algebra or an ordinary ring. Then*

$$\operatorname{gh. dim.} E = \operatorname{gh. dim.} E^{\operatorname{op}}.$$

*Proof.* The ghost dimension of  $E$  is the largest  $n$  such that there exists a right  $E$ -module  $X$  and a left  $E$ -module  $Y$  for which  $E_{n,*}^\infty$  is nonzero in the universal coefficient spectral sequence for  $\pi_*(X \wedge_E Y)$ . This is obviously left-right symmetric.  $\square$

Summing up, then, we are left with three possible definitions for the weak dimension of an  $S$ -algebra  $E$ . We list the basic inequalities between these definitions in the following theorem, which also proves the main conjecture of [HL08a] that  $\operatorname{gh. dim.} R = \operatorname{w. dim.} R$  for ordinary rings  $R$ .

**Theorem 2.10.** *Suppose  $E$  is an  $S$ -algebra or an ordinary ring. Then we have*

$$\begin{aligned} \operatorname{gh. dim.} E &= \sup_{X \text{ compact}} \operatorname{con. flat dim.} X = \sup_{X \text{ compact}} \operatorname{proj. dim.} X \\ &= \sup_{X \text{ compact}} \operatorname{flat dim.} X = \sup_{X \text{ arbitrary}} \operatorname{flat dim.} X. \end{aligned}$$

Furthermore,

$$\operatorname{gh. dim.} E \leq \sup_{X \text{ arbitrary}} \operatorname{con. flat dim.} X \leq \operatorname{w. dim.} E_*$$

with equality if  $E$  is an ordinary ring. Finally,

$$\operatorname{gh. dim.} E \leq \operatorname{Rouq. dim.} E$$

with equality if  $E_*$  is coherent.

*Proof.* The only thing we have not already proved is that equality holds in the first chain of inequalities when  $R$  is an ordinary ring. But the main result of [HL08a] is that  $\operatorname{w. dim.} R \leq \operatorname{gh. dim.} R$ , giving us the desired equalities.  $\square$

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